Math 4200
Friday November 20
4.4 Infinite series and infinite partial fractions

Announcements:

midterm problem $\cot z$ L.S. @ $z_0 = 0$

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \ldots$$

also
$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{4^2} + \cdots\right) = \frac{\pi^2}{3}$$

$$2\sum_{n=1}^{\infty} \frac{1}{n^4} = 2\frac{\pi^4}{90} = \frac{\pi^4}{45}$$

Is this a coincidence?
NO.

Today: 4.4
MW next week 5.1-5.2.
§4.4 HW appended. ~ due F next week.

4.4: Infinite series magic and infinite partial fractions, via contour integration....

Suppose we have an analytic $f(z)$, f analytic on $\mathbb{C} \setminus \{z_1, z_2, ..., z_k\}$. Suppose we wish to compute

$$\sum_{n=1}^{\infty} f(n).$$

Here's an approach that often works. For example, we'll see that it works in the cases of $f(z) = \frac{1}{z^k}$ where k is an even number, and gives closed form expressions for

$f(z) = \frac{1}{z^2}$ $f(z) = \frac{1}{z^4}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6}, \quad ....$$

Consider the auxillary function $g(z) = f(z)\pi \cot(\pi z)$. We choose to multiply by $\pi \cot(\pi z)$ partly because

is 1-periodic.

<u>Check</u>: $\pi \cot(\pi z)$ has simple poles precisely at each $n \in \mathbb{Z}$, with residue 1, so $f(z)\pi \cot(\pi z)$ has residue $f(n)$ when f is analytic at $z = n$.

$$f(z)\pi \cot(\pi z) = \frac{f(z)\ \pi \cos(\pi z)}{\sin(\pi z)}. \quad = \frac{g}{h}$$

$\sin \pi z$ has 1st zeros at integers

quotient has simple poles ($f(n) \neq 0$, f analytic; $f(n) = 0 \Rightarrow$ remov.)

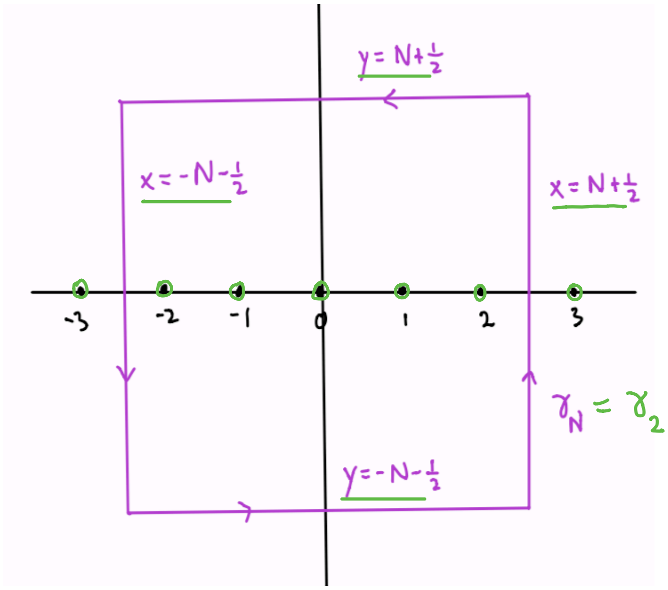
$$\mathrm{Res}\left(\frac{g}{h}; z_0\right) = \frac{g(z_0)}{h'(z_0)} \qquad h'(z) = \pi \cos \pi z$$

$$z_0 = n \qquad = \frac{f(n)\pi \cos \pi n}{\pi \cos \pi n} = f(n)$$

$g(z) = f(z)\,\pi \cot \pi z$

Now consider these special square contours γ_N, as $N \to \infty$. They are chosen so that $|\cot(\pi z)|$ is uniformly bounded on γ_N (by $M = 2$, for example), as $N \to \infty$.

appendix
use trig identities
$\dfrac{\cos \pi z}{\sin \pi z}$



<u>Theorem 1</u> Let $\{z_1, z_2, \ldots z_n\}$ be the singular points of f and suppose $|f(z)|$ decays sufficiently rapidly so that

$$\lim_{N \to \infty} \int_{\gamma_N} f(z)\,\pi \cot(\pi z)\,dz = 0 .$$

Then

$$\lim_{N \to \infty} \left(\sum_{\substack{j=-N \\ f \text{ analytic at } j}}^{N} f(j) \right) = -\left(\sum_{\substack{z_k \text{ singular} \\ \text{point of } f}} Res\big(f(z)\,\pi \cot(\pi z), z_k\big) \right).$$

proof:

Apply finite Residue Thm on γ_N, N large enough to contain $\{z_1, z_2, \ldots z_n\}$.

$$\int_{\gamma_N} f(z)\,\pi \cot(\pi z)\,dz = 2\pi i \sum_{\substack{\text{sing inside} \\ \gamma_N}} Res(\,\text{—}\,;\,).$$

$$= 2\pi i \left(\sum_{\substack{j=-N \\ f \text{ analytic } @ j}}^{N} f(j) + \sum_{z_k \text{ sing}} Res\big(f(z)\,\pi \cot(\pi z), z_k\big) \right)$$

as $N \to \infty$
$\div\, 2\pi i$

$$0 = \lim_{N \to \infty} \sum_{\substack{j=-N \\ f \text{ anal} @ j}}^{N} f(j) + \sum_{z_k \text{ sing fn } f} Res(\text{———})$$

Repeated for convenience: Theorem 1 Suppose $|f(z)|$ decays sufficiently rapidly so that

$$\lim_{N\to\infty} \int_{\gamma_N} f(z)\, \pi \cot(\pi z)\, dz = 0\ .$$

Then

$$\lim_{N\to\infty}\left(\sum_{\substack{j=-N\\ f\text{ analytic at } j}}^{N} f(j)\right) = -\left(\sum_{\substack{z_k \text{ singular}\\ \text{point of } f}} Res\big(f(z)\,\pi\cot(\pi z), z_k\big)\right).$$

f odd this sum is zero

Residue = 0

Example Find formulas for

$$\sum_{n=1}^{\infty}\frac{1}{n^2},\ \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$f(z)=\frac{1}{z^2}$

$\lim_{N\to\infty}\left(2\sum_{j=1}^{N}\frac{1}{j^2}\right) = -Res\left(\frac{1}{z^2}\pi\cot\pi z;\ 0\right)$

using $f(z)=\frac{1}{z^2}, \frac{1}{z^4}$. What goes wrong if you try to find a closed form expression for

2 $\quad f(z)=\frac{1}{z^3}$

$\therefore\ 2\sum_{n=1}^{\infty}\frac{1}{n^2} = -Res\left(\frac{1}{z^2}\pi\cot\pi z;\ 0\right)$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\ ?$$

Hint: Here's the beginning of the Laurent series for $\pi\cot(\pi z)$ at the origin:

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> series(π·cot(π·z), z=0, 12);
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$$z^{-1} - \frac{1}{3}\pi^2 z - \frac{1}{45}\pi^4 z^3 - \frac{2}{945}\pi^6 z^5 - \frac{1}{4725}\pi^8 z^7 - \frac{2}{93555}\pi^{10} z^9 - \frac{1382}{638512875}\pi^{12} z^{11} + O(z^{13}) \qquad \textbf{(1)}$$

$\frac{1}{z^2}$

$2\sum_{n=1}^{\infty}\frac{1}{n^2} = --\frac{\pi^2}{3}$

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$!

@0 $\quad \cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3$

$\Rightarrow\ \pi\cot\pi z = \pi\frac{1}{\pi z} - \pi\frac{1}{3}\pi z - \pi\frac{1}{45}\pi^3 z^3 - \dots$

$= \frac{1}{z} - \frac{1}{3}\pi^2 z - \frac{1}{45}\pi^4 z^3 - \dots$ ✓.

$2\sum_{n=1}^{\infty}\frac{1}{n^4} = -Res\left(\frac{1}{z^4}\pi\cot\pi z;\ 0\right)$

$f(z)=\frac{1}{z^4}$ $\qquad = \frac{\pi^4}{45}$.

$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

Theorem 2 If $f(z)$ is analytic on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_k\}$ and if for large $|z|$ there is an M for which $\left|f(z)\right| \le \dfrac{M}{|z|}$, then

$$\lim_{N \to \infty} \int_{\gamma_N} f(z)\, \pi \cot(\pi z)\, dz = 0. \quad !!$$

Thus, from Theorem 1,

$$\lim_{N \to \infty} \left(\sum_{\substack{j=-N \\ f \text{ analytic at } j}}^{N} f(j) \right) = -\left(\sum_{\substack{z_k \text{ singular} \\ \text{point of } f}} Res\big(f(z)\, \pi \cot(\pi z), z_k\big) \right)$$

proof. Let R be large enough so that all of the singularities of f have modulus less than R. Then f has a Laurent series in the complement of this disk, and because of the decay estimate it only has negative powers of z (because $f\left(\frac{1}{z}\right)$ has a removable singularity at $z=0$ and extends to equal zero there CHECK), !

$$f(z) = \frac{b_1}{z} + \sum_{m=2}^{\infty} \frac{b_m}{z^m} = \frac{b_1}{z} + g(z)$$

$f(z)$ analytic for $|z|>R$

$f\left(\frac{1}{z}\right)$ is analytic $\left|\frac{1}{z}\right|>R$

$0<|z|<\frac{1}{R}$

$\left|f\left(\frac{1}{z}\right)\right| \le \dfrac{M}{\left|\frac{1}{z}\right|}$ $\quad \left|\frac{1}{z}\right|>R$

$\left|f\left(\frac{1}{z}\right)\right| = M|z|$ for $0<|z|<\frac{1}{R}$

So $g(z) = f\left(\frac{1}{z}\right)$ analytic near 0

$f\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} a_n z^n$ near $z=0$

where there is a uniform estimate for g for $|z| \ge R$, $|g(z)| \le \dfrac{C}{|z|^2}$. Thus

$$\int_{\gamma_N} f(z)\, \pi \cot(\pi z)\, dz = \int_{\gamma_N} \underbrace{\frac{b_1}{z}}_{\text{odd}}\, \underbrace{\pi \cot(\pi z)}_{\text{odd}}\, \underbrace{dz}_{\text{odd}} + \int_{\gamma_N} g(z)\, \pi \cot(\pi z)\, dz.$$

The integrand in the first integral is even because it's a product of two odd functions but the contour differential dz is odd for this square contour, so the first integral evaluates to zero! And because $|\pi \cot(\pi z)| \le 2$ on the contours γ_N and $g(z)$ decays, the second integral's modulus can be estimated by

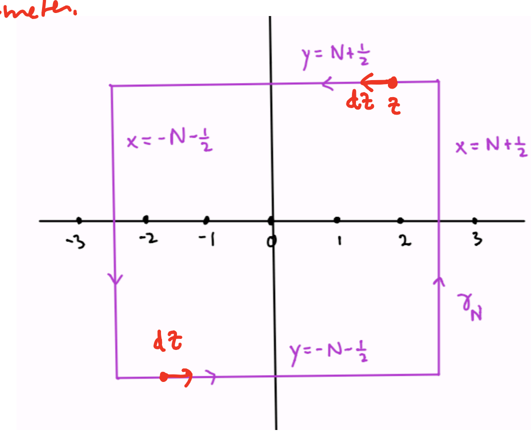
$f(w) = \sum_{n=1}^{\infty} a_n \frac{1}{w^n}$

$w = \frac{1}{z}$, $|w|>R$.

$$\left| \int_{\gamma_N} g(z)\, \pi \cot(\pi z)\, dz \right| \le \frac{2C}{N^2} \underbrace{4 \cdot (2N+1)}_{\text{perimeter.}} \to 0 \quad \text{as } N \to \infty.$$

$|g(z)| \le \dfrac{C}{|z|^2}$

you can check this!!



Repeated for reference: if for large z we have a decay estimate $\left|f(z)\right| \le \frac{M}{|z|}$ then

$$\lim_{N\to\infty}\left(\sum_{\substack{j=-N\\ f\text{ analytic at } j}}^{N} f(j)\right) = -\left(\sum_{\substack{z_k \text{ singular}\\ \text{point of } f}} Res\left(f(z)\,\pi\cot(\pi z), z_k\right)\right).$$

Examples 1) $f(z) = \frac{1}{z^{2k}}$, as we discussed in earlier example. Magic summation formulas for $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ based only on coefficients of the Laurent series for $\pi\cot(\pi z)$ at the origin!

Thm2 applies, for Thm1

2) For $z_0 \in \mathbb{C}\setminus\mathbb{Z}$, $f(z) = \frac{1}{z-z_0}$, which has a simple pole at z_0. So for z_0 not an integer,

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n-z_0} = -Res\left(\frac{1}{z-z_0}\,\pi\cot(\pi z), z_0\right) = -\pi\cot(\pi z_0).$$

So, replacing z_0 with z, multiplying the equation by -1, and arranging the sum on the left as a sum of two series that converge uniformly on compact subsets that avoid the integers:

$z_0 \to z$

$$\pi\cot(\pi z) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}$$

$n=0$

$$\pi\cot(\pi z) = \frac{1}{z} + \lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(\frac{1}{z-n}+\frac{1}{n}\right) + \sum_{n=1}^{N}\left(\frac{1}{z+n}-\frac{1}{n}\right)\right)$$

Because

$$\frac{1}{z-n}+\frac{1}{n} = \frac{z}{(z-n)n}, \qquad \frac{1}{z+n}-\frac{1}{n} = \frac{z}{(z+n)n}$$

each of the modified subseries converges uniformly on compact subsets that avoid the integers - by the Weirerstrass M test with comparison series the tail of

$$\sum_{n=N}^{\infty}\frac{1}{n^2}.$$

So

$$\pi\cot(\pi z) = \frac{1}{z} + \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{z-n}+\frac{1}{n}\right) + \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{z+n}-\frac{1}{n}\right).$$

This is like an infinite partial fractions decomposition for $\pi\cot(\pi z)$!

double poles @ all integers

3) In your homework you'll prove another infinite partial fractions expansion,

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

Notice that the pole locations and orders agree on both sides of the equation, so there would be some hope for the identity being true. There is a very quick proof of this identity if you can see it, based on doing something to the last identity on the previous page. An alternate proof would be to show that the difference of the two functions above has removable singularities at the integers; is $1-$ periodic in the real direction; and approaches zero uniformly in the imaginary direction, and then apply Liouville's Theorem to the difference.

These last two examples illstrate a general theory for infinite sum partial fraction expansions for *meromorphic* functions, i.e. functions which are analytic on $\mathbb{C}$ except for an at most countable set $\{z_k\}$ of isolated pole-type singularities (as opposed to essential singularities).

The *Mittag-Leffler Theorem* says you can basically create an infinite sum analytic function with prescribed isolated poles, and with prescribed negative power Laurent series at those poles. (There's a short page on this topic at Wikipedia that contains the two identities we've written down, as well as some others.)

Related to infinite sum formulas for Meromorphic functions, there are infinite product expansions as well for analytic functions, involving the zeroes of the analytic functions. (The connection between the two theories is the logarithm, which takes products to sums.) In Chapter 7 of our text there are infinite product identities related to the Riemann-Zeta function, for example.

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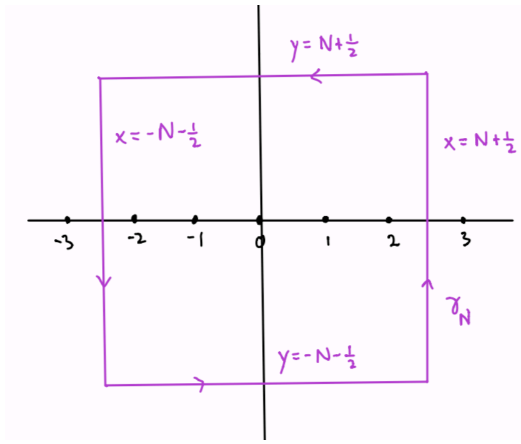
Appendix: Uniform bound estimates of $\cot(\pi z)$ on the half-integer contours γ_N .
These estimates hold:

- $|\cot(\pi z)| \leq 1$ on the vertical paths
- $|\cot(\pi z)| \leq 2$ on the horizontal paths (the bound limits to 1 as $N \to \infty$)



One efficient way to make these estimates is to use the various trig identities we discussed in Chapter 1. For $u, v \in \mathbb{R}$,

$$\cos(u+i\,v) = \cos(u)\cos(i\,v) - \sin(u)\sin(i\,v)$$
$$\cos(u+i\,v) = \cos(u)\cosh(v) - \sin(u)i\,\sinh(v)$$
$$\sin(u+i\,v) = \cos(u)\sin(i\,v) + \sin(u)\cos(i\,v)$$
$$\sin(u+\,i\,v) = \cos(u)i\,\sinh(v) + \sin(u)\cosh(v).$$

So,

$$\frac{\cos(u+i\,v)}{\sin(u+\,i\,v)} = \frac{\cos(u)\cosh(v) - \,i\,\sin(u)\sinh(v)}{\sin(u)\cosh(v) + \,i\,\cos(u)\,\sinh(v)}$$
$$\left|\frac{\cos(u+i\,v)}{\sin(u+\,i\,v)}\right|^2 = \frac{\cos^2(u)\cosh^2(v) + \sin^2(u)\sinh^2(v)}{\sin^2(u)\cosh^2(v) + \cos^2(u)\sinh^2(v)}.$$

$$\cos^2\left(\pm\pi\left(N+\frac{1}{2}\right)\right) = 0, \quad \sin^2\left(\pm\pi\left(N+\frac{1}{2}\right)\right) = 1$$
$$\Rightarrow \left|\frac{\cos(\pi\,x+i\,\pi\,y)}{\sin(\pi\,x+\,i\,\pi\,y)}\right|^2 = \frac{\sinh^2(\,\pi\,y)}{\cosh^2(\pi\,y)} \leq 1 \text{ on the vertical paths.}$$

And along the horizontal contours - for $v = \pm\pi\left(N+\frac{1}{2}\right)$, and using

$$\cosh^2(v) - \sinh^2(v) = 1,$$
$$\Rightarrow \left|\frac{\cos(u+i\,v)}{\sin(u+\,i\,v)}\right|^2 = \frac{\cos^2(u)\cosh^2(v) + \sin^2(u)\left(\cosh^2(v) - 1\right)}{\sin^2(u)\left(\sinh^2(v) + 1\right) + \cos^2(u)\sinh^2(v)}$$
$$= \frac{\cosh^2(v) - \sin^2(u)}{\sinh^2(v) + \sin^2(u)} \leq \frac{\cosh^2(v)}{\sinh^2(v)} \to 1 \quad \text{as } N \to \infty.$$

claim verified.

Math 4200-001
Week 13 concepts and homework
4.4
Due Friday November 27 at 11:59 p.m.

4.4: 2, 3, 4, 5, 8, 9 (number 9 won't actually be graded - consider it an extra credit challenge.)